Interaction of Ocean Waves and Currents: How Different Approaches may be Reconciled.

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ABSTRACT

The effect of ocean waves on near-surface currents is a topic about which interest has recently revived, as a result of various factors including the prediction of the drift of oil spills from marine accidents. In order to evaluate properly the effect of waves, it is necessary to employ a consistent formulation of the energy and momentum balance within the airflow, the wave field, and the water column. Although the usual Eulerian equations of fluid mechanics are often used, it is also very desirable to use a coordinate system which can represent vertical variations near the sea surface at scales smaller than the wave height: surface-following coordinate systems fall into this category. We review the application of such coordinate systems, and relate them to the Generalized Lagrangian Mean formalism of Andrews and McIntyre.

KEY WORDS: Ocean waves; wave–current interaction; wave refraction; Langmuir circulations; wave setup.

INTRODUCTION

The need to understand and forecast the state of the ocean, including the prediction of the drift of oil spills from marine accidents, is promoting the use of remote sensing techniques that measure surface velocities (High-Frequency radar and interferometric Synthetic Aperture Radar), together with the realistic numerical modeling of ocean currents. This context opens perspectives for the modeling of ocean waves and currents in a consistent way since both are important components of the jigsaw puzzle of ocean surface processes.

However, this task is complicated by the many interactions of waves and currents, and the specific theories that have been developed to address one or the other. Our intention is thus to reconcile the views of wave experts that have tended to follow Phillips (1977) and Bretherton and Garrett (1968) to describe the interaction of waves and the mean flow, and the views of surface mixing experts that have adopted the Craik-Leibovich (CL) theory (Leibovich 1983) to describe the generation of Langmuir circulations that are important for mixing in the upper ocean and arise from a coupling of waves with a vertically sheared current. That theory represents wave-mean flow coupling by the introduction of a uniform (at the very least non-divergent) and constant Stokes drift, it is thus a one-way forcing of the wave field driving the mean flow. It is quite successful in explaining Langmuir circulations (Leibovich 1983), but it does not reflect the feedback of the mean flow on waves, previously discussed by Garrett (1976). This produces a steady source of momentum and vorticity for the mean flow (e.g. Gjaja and Holm 1996), that also needs to be taken into account.

The vortex force that generates Langmuir circulations can be derived from the mean flow momentum equations that use radiation stresses in the fashion of Phillips (1977) for depth-integrated momentum equations, and more importantly how this force must be balanced, for the total flow by the refraction of waves over the surface current pattern. This leads to an interpretation of the Langmuir circulation as a wave-mean flow recoil mechanism, similar to the effect described by Bühler and McIntyre (2003).

In treating the problem, it is also useful to apply surface-following coordinates, in order to resolve mean flow variables at distances from the surface which are less than the wave height. A suitable method is the Generalized Lagrangian Mean (GLM) theory of Andrews and McIntyre (1978a), which can also be coupled to a version of the conservation law for wave action (Whitham 1967, Andrews and McIntyre 1978b). This can provide a practical parameterization of wave-mean flow coupling terms from wave spectra, accurate to second order in the wave slope, thereby extending Jenkins' (1989) results.

We finally discuss the implications our findings for the generation of Langmuir circulations and the general problem of coupled wave-ocean circulation modeling.

DEPTH-INTEGRATED EQUATIONS

We generally follow Hasselmann's (1971) notations we use dummy Greek indices α and β for horizontal components $x_1 = x$ and $x_2 = y$. Latin indices i and j refer to Eulerian coordinates $x_1 = x$, $x_2 = y$, and $x_3 = z$. We define ρ_a and ρ_w to be the densities of air and water respectively. We define p as the pressure P minus the hydrostatic equilibrium pressure $\int(-\rho g)\, dx_3$. Means may be time averages or averages over flow realizations.

The mean horizontal momentum $\overline{\mathbf{M}}$ is separated into a mean flow and a wave part,

$$\overline{\mathbf{M}} = \mathbf{M}^m + \mathbf{M}^w, \qquad (1)$$

with

$$M_\alpha^m = \overline{\int_{-h}^{\bar{\zeta}} \rho_w u_\alpha \mathrm{d}z}, \qquad (2)$$

and

$$M_\alpha^w = \overline{\int_{\overline{\zeta}}^{\zeta} \rho_w u_\alpha \mathrm{d}z}, \tag{3}$$

where $\zeta(x,y)$ is the position of the free surface, (u_x, u_y) is the horizontal velocity vector, and the overbar denotes the averaging operator.

The present derivation is a simple extension of Ardhuin et al. [2003, manuscript submitted to Journal of Physical Oceanography], where a mean current is added, which is an extension of Garrett's (1976) equations, to which the Coriolis force is added.

The horizontal mean flow momentum equation is then

$$\begin{aligned} \frac{\partial \mathbf{M}^m}{\partial t} &= \left[\nabla\cdot\tau^m - f_3\mathbf{e}_3\times\mathbf{M}^m + \overline{p^a}\nabla\overline{\zeta} + (p^m+gh)_{-h}\nabla h \right. \\ &\left. +\mathbf{T}^a - \mathbf{T}^b\right] + p_{-h}^w\nabla h - f_3\mathbf{e}_3\times\mathbf{M}^w - \nabla\cdot\mathbf{S} \\ &-\nabla\cdot\tau^{\mathrm{rad2}} - \frac{\partial\mathbf{M}^w}{\partial t}, \end{aligned} \tag{4}$$

where τ^m is a horizontal tensor that contains mean momentum advection terms and mean-flow pressure gradients (including hydrostatic pressure) and viscous stresses,

$$\tau_{\alpha\beta}^m = -\int_{-h}^{\overline{\zeta}} \rho_w\left(\overline{u}_\alpha\overline{u}_\beta\right) + \delta_{\alpha\beta}\left(p^m - \rho_w g z\right) + \mu_w\frac{\partial^2\overline{u}_1}{\partial x_\beta\partial x_\beta}\mathrm{d}x_3, \tag{5}$$

$\mathbf{T}^a$ is the wind stress vector, equal to the total atmosphere to ocean momentum flux, and $\mathbf{T}^b$ is the bottom stress vector, equal to the total ocean to bottom momentum flux. These two fluxes are counted positive downwards. The last five terms in (4) represent wave effects on the mean flow that are not represented in current ocean-circulation models. $p_{-h}^w\frac{\partial h}{\partial x_\alpha}$ can be neglected in deep water or for mild bottom slopes and will not be considered here. In steady quasi-geostrophic conditions, the divergence of the Hasselmann stress $\partial\mathbf{T}^H/\partial x_3 = -\mathbf{f}\times\mathbf{M}^w$, will drive a mean Eulerian transport that will exactly balance the Stokes drift, giving a zero Lagrangian wave-induced transport. In other conditions, such as variations in time of the wave field, the Lagrangian wave-induced transport may not be balanced and waves may drive net mass transports (Hasselmann 1970).

We have chosen here to use the usual notation $S_{\alpha\beta}$ for the radiation stresses in the absence of a mean current. The tensor S can be computed from the wave elevation variance spectrum F, to second order in the wave slope,

$$S_{\alpha\beta} = \rho_w g\int_{\mathbf{k}} F(\mathbf{k})\left[\left(\frac{C_g}{C} - \frac{1}{2}\right)\delta_{\alpha\beta} + \frac{C_g}{C}k_\alpha k_\beta/k^2\right]\mathrm{d}\mathbf{k} \tag{6}$$

with C and C_g the phase and group speed velocities, respectively.

The term τ^{rad2} represents the advection of wave momentum by the mean flow,

$$\tau_{\alpha\beta}^{rad2} = U_\alpha M_\beta^w + U_\beta M_\alpha^w. \tag{7}$$

Following Garrett (1976), vortex force appears by using the equation for the conservation of the wave action. We shall keep here the possibility that the action evolves with a source term S_{tot}/σ,

$$\frac{\partial(E/\sigma)}{\partial t} + \nabla\cdot\left[(U+C_g)E/\sigma\right] = \frac{S_{\mathrm{tot}}}{\sigma}. \tag{8}$$

This equation for monochromatic waves can be seen as the limit for small $\Delta\mathbf{k}$ of it spectral counterpart, generally used in wave models, and here integrated over wavenumbers,

$$\begin{aligned} \frac{\partial\int_{\mathbf{k}}(F(\mathbf{k})/\sigma)\mathrm{d}\mathbf{k}}{\partial t} + \nabla\int_{\mathbf{k}}\cdot\left[(U+C_g)F(\mathbf{k})/\sigma\right]\mathrm{d}\mathbf{k} \\ = \int_{\mathbf{k}}\frac{S_{tot}(\mathbf{k})}{\sigma}\mathrm{d}\mathbf{k}. \end{aligned} \tag{9}$$

Using the time rate of change of the wave number $\mathbf{k}$,

$$\frac{\partial k_\alpha}{\partial t} + (U+C_g)\frac{\partial k_\alpha}{\partial x_\beta} = -k_\beta\frac{\partial U_\beta}{\partial x_\alpha}, \tag{10}$$

and the fact that $M_\alpha^w = \int_{\mathbf{k}} k_\alpha F(\mathbf{k})/\sigma\mathrm{d}\mathbf{k}$, we find

$$\begin{aligned} \frac{\partial M_\alpha^w}{\partial t} &= -\frac{\partial}{\partial x_\beta}\left[M_\alpha^w\left(U_\beta + C_g\frac{k_\beta}{k}\right)\right] \\ &- M_\beta^w\frac{\partial U_\beta}{\partial x_\alpha} + \int_{\mathbf{k}}\frac{k_\alpha S_{tot}(\mathbf{k})}{\sigma}\mathrm{d}\mathbf{k}. \end{aligned} \tag{11}$$

This equation can now be replaced in (4) to give

$$\begin{aligned} \frac{\partial\mathbf{M}^m}{\partial t} &= \left[\nabla\cdot\tau^m - f_3\mathbf{e}_3\times\mathbf{M}^m + \overline{p^a}\nabla\overline{\zeta} + (p^m+gh)_{-h}\nabla h \right. \\ &\left. +\mathbf{T}^a - \mathbf{T}^b\right] + \mathbf{F}^m, \end{aligned} \tag{12}$$

with the wave force $\mathbf{F}^m$ taking the form

$$\begin{aligned} F_\alpha^m &= \left[\mathbf{M}^w\times(\Omega+f_3)\mathbf{e}_3\right]_\alpha - \int_{\mathbf{k}}\frac{k_\alpha S_{tot}(\mathbf{k})}{\sigma}\mathrm{d}\mathbf{k} \\ &- U_\alpha\frac{\partial}{\partial x_\beta}M_\beta^w - \frac{\partial}{\partial x_\alpha}\Pi_\star, \end{aligned} \tag{13}$$

with Ω the vertical component of vorticity, assumed vertically uniform, and

$$\Pi_\star = \int_{\mathbf{k}}\left[\left(\frac{C_g}{C} - \frac{1}{2}\right)F(\mathbf{k})\right]\mathrm{d}\mathbf{k}. \tag{14}$$

The continuity equation is given by Hasselmann (1971) and Garrett (1976),

$$\rho_w\frac{\partial\overline{\zeta}}{\partial t} + \rho_w\frac{\partial}{\partial x_\alpha}\left[U_\alpha\left(h+\overline{\zeta}\right)\right] = -\frac{\partial M_\alpha^w}{\partial x_\alpha}. \tag{15}$$

These equation must be compared to the Craik-Leibovich (CL) equations for vertically uniform currents. The momentum equation (13) is a small extension of Garrett's (1976) equation (3.11), written in the form of Craik and Leibovich. The vortex force term that drives Langmuir circulations, $\mathbf{M}^w\times(\Omega+f_3)\mathbf{e}_3$, and includes the planetary vorticity f_3 is common to both equations. The $\Pi_\star$ term here is the vertically integrated wave-induced pressure gradient and should be part of the modified pressure term in Craik-Leibovich equations, that is usually written as $\varpi = \left|\mathbf{u}+\mathbf{u}^{\mathrm{st}}\right|^2 - |\mathbf{u}|$, where $\mathbf{u}^{\mathrm{st}}$ is the Stokes drift and $\mathbf{u}$ is the horizontal Eulerian velocity vector. However, the Craik-Leibovich theory assumes that the Stokes drift is non-divergent and the resulting definition of the mean pressure is affected by this definition (see Holm 1996).

It is therefore not clear that the depth-integrated equations of motion are the same, to second order in the wave slope, when horizontal variations in the wave field are allowed.

Our equations also suggest that it may be desirable to include the vertical variation of currents to generalize the radiation stress approach followed here. However, shear at the surface is generally weak (e.g. Smyth et al. 2002), and should only affect the wave field to a limited extent. More importantly, the radiation stress approach shows that the 'vortex force' compensates for the refraction force exerted on the wave field. Thus the existence of a vortex force in a Langmuir circulation system must be accompanied by wave refraction over the surface current pattern. In this context it is likely that the current pattern creates a wave field pattern, with gradients that should be taken into account. This coupling also offers one more mechanism for wave scattering. Since Craik-Leibovich theory as such assumes a uniform wave field, a more general framework must be sought. This does not invalidate the explanation of Langmuir circulations by Craik-Leibovich theory, but it may modify the structure and strength of the wave forcing.

SURFACE-FOLLOWING COORDINATE FORMULATIONS

A more general set of equations can be obtained by removing the integration over depth, using the same approach with Eulerian coordinates and a small surface slope expansion of wave variables. However, this type of approach fails above the wave trough level because a fixed volume element is successively filled by air and water, requiring a two-phase approach. A number of different approaches, involving changes of variables, have been suggested to remove this difficulty: a sigma-coordinate transformation (Mellor 2003); the generalized Lagrangian mean (GLM) formulation (Andrews and McIntyre 1978a); a limited time Lagrangian approach (Pierson 1962, Weber 1983, Jenkins 1987). Similar surface-following coordinate approaches have been employed for airflow over waves (Brooke Benjamin 1959, Jenkins 1992, Miles 1996).

The GLM formulation, in which the mean current velocity ($\overline{\mathbf{u}}^L$) is equal to the mean velocity of fluid particles, is perhaps the approach which is the most dynamically satisfactory. In fact, the Lagrangian averaging technique employed in the GLM formulation has been shown to preserve the variational structure of the Euler-Poincaré framework for fluid dynamics (Holm 2002). It is necessary to bear in mind some disadvantages of the technique. The equations derived are rather complex, although not necessarily more so than other methods involving curvilinear coordinates. A pure Lagrangian coordinate formulation has a more direct interpretation, with a coordinate transformation jacobian which is constant for incompressible flow, although it cannot be applied for long times due to the distortion of the coordinate system. A sigma-coordinate method, in which the coordinates move only vertically, leads to a simpler interpretation of vertical fluxes, since there is no variation of the area of two-dimensional cross-sections. If, as is the case for breaking waves and in atmospheric flow over waves, there exist critical layers where the mean flow velocity is equal to the wave phase speed, the amplitude of the oscillatory part of the coordinate transformation tends to infinity and the GLM method breaks down, so that a more general curvilinear coordinate formulation becomes necessary. It should also be noted that the GLM coordinate system is not completely surface-following: the mean equations are with respect to a fixed Cartesian coordinate system, and the air-water interface has a vertical coordinate equal to the Lagrangian mean vertical surface displacement $\overline{\zeta}^L$, a quantity which is of second order in wave slope for small-amplitude surface waves, and will vary in space and time (Grimshaw 1984, Groeneweg and Klopman 1998).

Nevertheless, the GLM technique has provided a very powerful foundation for the analysis of wave–mean flow interaction. It provides a convenient framework point for modeling wave–current interaction in a consistent manner, particularly since related equations for the evolution and propagation of waves, in terms of wave action conservation, have also been formulated (Andrews and McIntyre 1978b).

GLM foundation for wave–current coupling

The following set of equations is based on those of Andrews and McIntyre (1978a;b)—see also Craik (1985):

$$\overline{D}^L(\overline{u}_i^L - \mathsf{p}_i) + \overline{u}_{k,i}^L(\overline{u}_k^L - \mathsf{p}_k) + (\mathbf{f}\times\overline{\mathbf{u}}^L)_i + \overline{\Pi}_{,i}^L = -\overline{X}_i^L - \overline{\xi_{k,i}X_k^l}, \tag{16}$$

$$\Pi_{,i} \equiv F_i + \frac{\partial}{\partial x_i}\int \frac{dP}{\rho},$$

$$\overline{D}^L\tilde{\rho} + \tilde{\rho}\nabla\cdot\overline{\mathbf{u}}^L = 0, \tag{17}$$

$$\overline{D}^L \equiv \partial/\partial t + \overline{\mathbf{u}}^L\cdot\nabla,$$

$$\overline{D}^L A^\eta + \tilde{\rho}^{-1}\nabla\cdot\mathbf{B}^\eta = \mathcal{H}. \tag{18}$$

In the above equations, we employ the notation $(\,\cdot\,)_{,i}$ for partial differentiation in the ith coordinate direction. The pressure P does not include the hydrostatic correction of the previous sections.

The generalized Lagrangian mean $\overline{\phi}^L$ of a dependent variable ϕ is defined in terms of the particle displacement $\xi(\mathbf{x},t)$ due to the wave motions, as follows:

$$\overline{\phi(\mathbf{x},t)}^L = \overline{\phi^\xi(\mathbf{x},t)}, \qquad \phi^\xi(\mathbf{x},t) = \phi(\mathbf{x}+\xi,t), \tag{19}$$

with $\overline{(\,\cdot\,)}$ being a suitable averaging operator. The generalized Lagrangian mean (GLM) velocity $\overline{\mathbf{u}}^L$ is defined by

$$\left(\frac{\partial}{\partial t} + \overline{\mathbf{u}}^L\cdot\nabla\right)[\mathbf{x}+\xi(\mathbf{x},t)] = \mathbf{u}(\mathbf{x}+\xi,t). \tag{20}$$

We also define $(\,\cdot\,)^l = (\,\cdot\,)^\xi - \overline{(\,\cdot\,)}^L$.

In Eq. 16, $\mathsf{p}_i = -\overline{\xi_{j,i}[u_j^l + (\frac{1}{2}\mathbf{f}\times\xi)_j]}$ is the wave pseudomomentum per unit mass, $\mathbf{f}$ is the planetary vorticity vector, F_i represents external body forces, and X_i represents dissipative forces. In (17),

$$\tilde{\rho} = \rho^\xi J, \qquad J = \det\{\delta_{ij} + \xi_{i,j}\}, \tag{21}$$

J being the Jacobian determinant of the mapping $\mathbf{x} \mapsto \mathbf{x}+\xi$.

In Eq. 18, $A^\eta = \overline{\xi_{j,\eta}[u_j^l + (\frac{1}{2}\mathbf{f}\times\xi)_j]}$ is the generalized wave action density, and $B_i^\eta = \overline{P^\xi\xi_{j,\eta}K_{ji}}$, where K_{ji} is the (j,i)th co-factor of the Jacobian J in (21). The averaging is in this case with respect to the label η, for which various alternatives can be chosen.

In the case of waves propagating in the x_1-direction, we may choose $\eta = -x_1$ (Craik 1985), and A^η becomes A^{-x_1}, the x_1-component of wave pseudomomentum. We may also choose $\eta = t - \mathbf{k}\cdot\mathbf{U}/\sigma$ (see below), for which $A^\eta = A^t$ is now the wave energy density, or $\eta = -\theta$, θ being the wave phase, for which $A^{-\theta}$ is the wave action density. The relations (x, momentum), (time, energy), and (phase, action), are the same as the relation between translational symmetry with respect to a coordinate, and the related conserved mechanical quantity, as specified in Noether's Theorem.

For adiabatic or constant-density flow, $\mathcal{H}$ is given by Andrews and McIntyre (1978b) as

$$\mathcal{H} = -\overline{\xi_{i,\eta} X_i^l}. \tag{22}$$

If the density is constant and the only body force is gravity, expressed by a potential Φ, we have in in Eq. 16, from Andrews and McIntyre (1978a):

$$\overline{\Pi}_{,i}^L = \overline{P}^L + \overline{\Phi}_{,i}^L - \left\{ \overline{u_j^\xi \left[\tfrac{1}{2} u_j^\xi + (\tfrac{1}{2}\mathbf{f}\times\xi)_j \right]} \right\}_{,i}. \tag{23}$$

We may also assume that the forces X_i, which may represent viscosity or the averaged effects of smaller-scale turbulent eddies, conserve momentum locally, and so may be expressed as the divergence of a stress tensor. In a fixed coordinate system, $X_i = \partial\tau_{ij}/\partial x_j$, which becomes in the GLM representation (cf. Jenkins 1992), we have

$$X_i^\xi = (\tau_{il}^\xi K_{lj})_{,j}. \tag{24}$$

Surface gravity waves, without friction or rotation.

We first assume that the wave field consists of surface gravity waves, on a fluid of depth h, and neglect frictional and Coriolis forces. Since the waves are restricted to propagating parallel to the surface, the relevant quantities for wave energy, action, and pseudomomentum are those obtained by integrating with respect to the vertical coordinate.

We consider firstly a single wave component, with amplitude a, phase θ, wavenumber $\mathbf{k} = \nabla\theta$, and intrinsic angular frequency $\sigma = -\theta_{,t} + \mathbf{k}\cdot\mathbf{U}$, the dispersion relation being $\sigma^2 = gk\tanh kh$, where $k = |\mathbf{k}|$. We have, to $O(\epsilon)$,

$$\begin{aligned} \xi_\alpha &= \Re\left\{ ia\frac{k_\alpha}{k}\frac{e^{i\theta}\cosh(k(x_3+h))}{\sinh kh} \right\}, \\ \xi_3 &= \Re\left\{ a\frac{e^{i\theta}\sinh(k(x_3+h))}{\sinh kh} \right\}, \end{aligned} \tag{25}$$

$$\begin{aligned} u_\alpha^l &= \Re\left\{ a\frac{k_\alpha}{k}\sigma\frac{e^{i\theta}\cosh(k(x_3+h))}{\sinh kh} \right\}, \\ u_3^l &= \Re\left\{ -ia\sigma\frac{e^{i\theta}\sinh(k(x_3+h))}{\sinh kh} \right\}, \end{aligned} \tag{26}$$

$$P^l = -\rho g\xi_3 + \Re\left\{ a\frac{\sigma^2}{k}\frac{e^{i\theta}\cosh(k(x_3+h))}{\sinh kh} \right\}. \tag{27}$$

The "wave advection" velocity $\mathbf{U} = (U_1, U_2)$ is given approximately by

$$U_\alpha = 2k\int_{-\infty}^0 \overline{u}_\alpha^L(x_3)\, e^{2kx_3}\, dx_3 \tag{28}$$

(Teague 1986).

The most convenient definition of A^η uses $\eta = -\theta$, so that, to $O(\epsilon^2)$,

$$\int_{-h}^0 \rho A^{-\theta}(x_3)\, dx_3 = \frac{1}{2}\frac{\rho g a^2}{\sigma} = \frac{E}{\sigma}, \tag{29}$$

the usual definition of wave action. If we assume the waves to propagate in the x_1-direction, the non-advective wave action flux $\mathbf{B}^{-\theta}$ is given by:

$$\begin{aligned} B_1^{-\theta} &= -\overline{P^\xi\xi_{1,\theta}K_{11}} - \overline{P^\xi\xi_{3,1}K_{31}} \\ B_3^{-\theta} &= -\overline{P^\xi\xi_{1,\theta}K_{13}} - \overline{P^\xi\xi_{3,1}K_{33}} \end{aligned} \tag{30}$$

A noteworthy feature of (30) is that the surface pressure-slope covariance, proportional to $k_\alpha\overline{P^\xi\xi_{3,\alpha}}$, gives a negative $O(\epsilon^2)$ contribution to the vertical wave action flux, as is required when waves are generated by the Jeffreys–Miles mechanism. Andrews and McIntyre (1978b) noted that, to $O(\epsilon^2)$, B_i^η was equal to $\overline{P'\xi_{i,\eta}}$ plus a term with zero divergence, where P' is the Eulerian pressure fluctuation.

Radiation stress and pseudomomentum flux.

Equation 18 may be re-written as

$$(\tilde\rho A^\eta)_{,t} + \nabla\cdot(\overline{\mathbf{u}}^L\tilde\rho A^\eta + \mathbf{B}^\eta) \tag{31}$$

(Andrews and McIntyre 1978b, Eqs. 2.15–16). To $O(\epsilon^2)$, neglecting advection by the mean current,

$$\int_{-h}^0 \mathbf{B}^\eta(x_3)\, dx_3 = \rho\mathbf{v}_g\int_{-h}^0 A^\eta(x_3)\, dx_3, \tag{32}$$

where $\mathbf{v}_g = \partial\sigma/\partial\mathbf{k}$ is the wave group velocity. It can thus be seen that the waves transport wave energy and pseudomomentum, as well as wave action, at the same speed as the wave action.

The quantity $\mathbf{B}^{-\mathbf{x}}$ can be regarded as a radiation stress. For waves propagating in the x_1-direction, to $O(\epsilon^2)$ in the absence of a mean current, when integrated vertically it becomes

$$R_{11} = \int_{-h}^0 \mathbf{B}_1^{-\mathbf{x}}(x_3)\, dx_3 = E\left(\frac{1}{2}\frac{\sigma}{k} + \frac{\sigma h}{\sinh 2kh}\right) = Ev_a g. \tag{33}$$

For finite water depth, this is different from the radiation stress computed in a fixed, Eulerian coordinate system (Longuet-Higgins and Stewart 1962):

$$\begin{aligned} S_{11} &= \overline{\int_{-h}^{\zeta}(P + \rho u_1^2)\, dx_3} - \int_{-h}^{\overline{\zeta}}\rho g(\overline{\zeta} - x_3)\, dx_3 \\ &= E\left(\frac{1}{2}\frac{\sigma}{k} + \frac{2\sigma h}{\sinh 2kh}\right). \end{aligned} \tag{34}$$

However, the result (34) is also obtained in the GLM coordinate system, using the $O(\epsilon^2)$ formula

$$\begin{aligned} S_{11} &= \overline{\int_{-h}^0 \left(P^l + (u_1^l)^2\right)\left(1 + \xi_{3,3}\right) dx_3} \\ &= \int_{-h}^0 \left(\overline{P^l\xi_{3,3}} + \overline{(u_1^l)^2}\right) dx_3. \end{aligned} \tag{35}$$

The difference between the momentum flux S_{11} and the pseudomomentum flux R_{11} is accounted for by an $O(\epsilon^2)$ change in the (Eulerian) mean surface elevation (Longuet-Higgins and Stewart 1962):

$$\overline{\zeta} = -\frac{1}{2}\frac{a^2 k}{\sinh 2kh}, \tag{36}$$

which corresponds to the mean Bernoulli pressure reduction at the sea bottom due to the oscillatory wave motion, $\rho\overline{[u_1^l(-h)]^2}$. Horizontal depth variations thus induce mean hydrostatic pressure gradients which account for the variations in $S_{11} - R_{11}$. Note that the GLM surface elevation will in this case become

$$\overline{\zeta}^L = -\frac{1}{2}\frac{a^2 k}{\sinh 2kh} + \frac{1}{2}\frac{a^2 k}{\tanh kh}, \tag{37}$$

the extra term $(\frac{1}{2}a^2 k/\tanh kh)$ being required as a result of mass conservation.

Note that Eq. 37 is only valid in the absence of dissipative processes. If waves break when they propagate to the shore, the momentum which they then transfer to the mean current will, of course, cause an *increase* in the mean surface elevation.

Effect of dissipative processes.

If we apply Eq. 24 to the GLM equations (16–23), we obtain a system in which the dissipative forces cannot change the total momentum of the system except at the surface and bottom boundaries. A systematic solution of the GLM equations in such a system, with $\mathbf{f} = 0$ has been given by Groeneweg and Klopman (1998).

For non-zero $\mathbf{f}$, where the dissipative forces are the result of a constant eddy viscosity ν, Weber (1983) and Jenkins (1986) employed an $O(\epsilon^2)$ perturbation expansion in Lagrangian coordinates to derive equations for the mean current in deep water. An important feature of the solution is a surface viscous boundary layer (vorticity layer) of thickness $O((2\nu/\sigma)^{1/2})$, which may be regarded as altering the surface boundary conditions for the fluid underneath. If wave dissipation is due to this constant eddy viscosity, the wave pseudomomentum is converted to momentum of the mean Eulerian current in this boundary layer.

Jenkins (1987) used the same technique with a vertically-varying eddy viscosity, and found out that, in addition to the conversion of wave pseudomomentum to momentum of the mean current in the vorticity layer, a proportion of the wave pseudomomentum also acted as a momentum source for the mean current within the water column, the source strength being proportional to the product of the eddy viscosity gradient and e^{2kx_3} (in deep water). The waves tend to decay according to

$$a \propto \exp\left[-2k^2\left(\int_{-\infty}^{0} 2k\,\nu(x_3)\,e^{2kx_3}\,dx_3\right)t\right].$$

One can simulate wave dissipation by wave breaking and white-capping (e.g. Hasselmann 1974), by employing a vertically-varying eddy viscosity which has the same wave-frequency-dependent wave-damping effect. It is impossible to use the same eddy viscosity to damp the wave energy as one uses for the diffusion of momentum within the current field: the former must be much smaller than the latter. This is because of the fact that the current will be affected by turbulent eddies and other motions such as Langmuir circulations, which have time scales too great to cause much wave damping. It is therefore necessary to employ timescale-dependent eddy-viscosity profiles in order to use this approach, as was done by Jenkins (1989). The wave model results also had to be adjusted to ensure that wave energy, momentum, and action were properly conserved. In Eq. 16, we then have, for deep water,

$$\begin{aligned} -\overline{X}_i^L - \overline{\xi_{k,i}X_k^l} \;&=\; [\nu(\overline{\mathbf{u}}^L - \mathsf{p})_{,3}]_{,3} \\ &\quad - \int_{\mathbf{k}} 2k\,N(\mathbf{k},x_3)\,e^{2kx_3}\,\mathbf{k}S_{\mathrm{ds}}(\mathbf{k}); \end{aligned} \tag{38}$$

$$\int_{-\infty}^{0} 2kN(\mathbf{k},x_3)\,e^{2kx_3}\,dx_3 = 1,$$

where $-S_{\mathrm{ds}}(k)$ is the rate of wave action dissipation, and $2k\,N(\mathbf{k},x_3)\,e^{2kx_3}$ represents how the dissipation is distributed with depth. If the wave dissipation can be regarded as being due to an eddy viscosity ν_w, then $N(\mathbf{k},x_3)$ is proportional to ν_w. The boundary condition at the water surface is

$$\nu(\overline{\mathbf{u}}^L - \mathsf{p})_{,3} = \frac{\tau}{\rho} - \int_{\mathbf{k}} \mathbf{k}S_{\mathrm{in}}(k), \tag{39}$$

where τ is the total applied surface stress, and $S_{\mathrm{in}}(k)$, and $S_{\mathrm{in}}(k)$ is the rate at which wave action is applied to the wave field at the sea surface, by means of pressure or shear stress fluctuations.

Vorticity equation, and wave—mean flow recoil

Equations which describe the evolution of the vorticity field under the influence of the Stokes drift (see Eq. 43 below), which may lead to the development and maintenance of Langmuir circulations, were derived by Craik and Leibovich (1976), on the basis of a rather complex perturbation analysis in Eulerian coordinates. This analysis becomes simpler if one employs the GLM formulation.

Assuming that the dissipative forces are the result of a constant viscosity or eddy viscosity ν

$$-X_i = \nu\nabla^2 u_i \tag{40}$$

in fixed Cartesian coordinates, Leibovich (1980) derived the following equations for a constant-density fluid, with relative error $O(\mathcal{U}/c) + O(\epsilon^2\nu/(\mathcal{U}\lambda)) + O(|\mathbf{f}|/\sigma)$:

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \overline{\mathbf{u}}^E\cdot\nabla\right)\overline{\mathbf{u}}^E + \mathbf{f}\times(\overline{\mathbf{u}}^E + \overline{\mathbf{u}}^S) + \nabla(\overline{\Pi}^L + \overline{\mathbf{u}}^E\cdot\overline{\mathbf{u}}^S) \\ = \overline{\mathbf{u}}^S\times\mathrm{curl}\,\overline{\mathbf{u}}^E + \nu\nabla^2\overline{\mathbf{u}}^E, \end{aligned} \tag{41}$$

$$\nabla\cdot\overline{\mathbf{u}}^E = 0, \tag{42}$$

where $\overline{\mathbf{u}}^E = \overline{\mathbf{u}}^L - \mathsf{p}$ is, to $O(\epsilon^3\mathcal{U})$, the Eulerian mean velocity (or "quasi-Eulerian" velocity (Jenkins 1986), since it is referred to mean particle positions rather than fixed points in space), $\overline{\mathbf{u}}^S = \overline{\mathbf{u}}^L - \overline{\mathbf{u}}^E, \mathcal{U}$ being a typical mean velocity, and ϵ a typical ratio of particle excursion ξ to wavelength λ. For $\mathbf{f} = 0$, Eqs. 41–42 may be reduced to the CL equations

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\overline{\mathbf{\Omega}} = (\overline{\mathbf{\Omega}}\cdot\nabla)(\overline{\mathbf{u}}^E + \overline{\mathbf{u}}^S) - (\overline{\mathbf{u}}^E + \overline{\mathbf{u}}^S)\cdot\nabla\overline{\mathbf{\Omega}}, \tag{43}$$

where $\overline{\mathbf{\Omega}}$ is the wave-averaged vorticity vector (Craik and Leibovich 1976, Leibovich 1977, Craik 1985).

Holm (1996) provides an interesting treatment of the CL equations. He derives the inviscid form of the equations by means of Hamilton's principle, with a Lagrangian

$$\mathcal{L} = \int\int \left[\frac{1}{2}D|\overline{\mathbf{u}}^L|^2 - D\overline{\mathbf{u}}^L\cdot\overline{\mathbf{u}}^S - P(D-1)\right] d^3x\,dt, \qquad (44)$$

where D is the determinant of $\nabla\mathbf{y}$, the (y_1, y_2, y_3) being Lagrangian particle labels, and the P (the pressure) is a Lagrange multiplier. The inviscid CL equations are obtained by requiring $\mathcal{L}$ to be stationary with respect to variations in $\mathbf{y}$.

Refraction and wave–mean flow recoil. The refraction of waves in a rotational mean current field has been shown by Bühler and McIntyre (2003) to exert a force on the mean flow at locations, such as the vortex core, where the wave field is in fact not present. This fundamentally new wave–mean interaction effect is nevertheless capable of being reproduced within the system of equations (16–23), for example, in numerical coupled wave—ocean circulation models. The Bühler–McIntyre mechanism is non-dissipative, so conserves total energy, action, and momentum will be conserved, and the propagation of waves of small amplitude under the influence of the mean current can be computed by ray tracing or the WKBJ approximation. Any change in the wave pseudomomentum due to refraction by the current will necessarily cause a corresponding reaction (acceleration) of the mean flow.

An application of the technique outlined here, with the addition of dissipative mechanisms, may enable us to put on a quantitative footing the wave refraction mechanism for generation and maintenance of Langmuir circulations (Garrett 1976), with the addition of a full three-dimensional description of the mean and fluctuating flow field.

DISCUSSION AND CONCLUSIONS

Garrett proposed that waves would be focused in the areas of maximum current velocity. He then went on to discuss the generation of Langmuir circulations as a consequence of a feedback with preferential wave breaking in these areas. Without going that far, it is apparent that such spanwise gradients in the wave field act as an extra force on the mean flow (this is done through the gradient of the modified pressure in Craik–Leibovich theory). Use of that theory is limited because of the hypothesis of a divergence-free Stokes drift, which is generally not the case. In this paper we give pointers to the use of more general theories such as the Generalized Lagrangian Mean to model or study real conditions.

However, the current-induced relative modification of the wave momentum should be of order $\delta M^w/M^w = \delta U/C$ with δU the difference of velocity between the Langmuir convergence and divergence zones. The relative change in the vortex force should then be of order $\delta U/C$ which should not exceed 10 %. If waves are indeed larger in the convergence zone, the vortex force term becomes larger while the mean wave-added pressure acts to increase the vortex force. Besides, the term $-U_\alpha \frac{\partial}{\partial x_\beta} M^w_\beta$ in 13 would then act as a force in the downwind direction, strengthening the jets in the convergence areas and weakening the currents in the divergence area Therefore it may be anticipated that a varying wave field would slightly increase the asymmetry of the Langmuir circulations and enhance variation in downwind velocity between convergence and divergence zones.

Although small-scale studies may use any type of formalism, application to large scale modeling is more easily performed by adapting existing tools, as proposed by Jenkins (1987; 1989). These are primitive equation models with coordinates that follow a mean free surface and many possible turbulence closure schemes, and the spectral wave models. The GLM approach summarized here, although it gives rise to a rather complex set of equations, and although its results must be interpreted carefully as a result of its non-constant coordinate transformation Jacobian determinant, provides a good starting point, since it treats the various conserved dynamical quantities such as energy, momentum, wave action, and wave pseudomomentum, in a consistent manner.

ACKNOWLEDGMENTS

We are grateful to Tanos Elfouhaily for discussions on several coordinate choices and the physical signification of different terms helped focus the present work.

Alastair Jenkins acknowledges support from the Research Council of Norway under Project No. 155923/700.

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